Cohomology of Drinfeld Modular Varieties, Part II

Automorphic forms, trace formulas and Langlands correspondence

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with an appendix by
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Preface

The second volume of *Drinfeld modular varieties* is devoted to the Arthur–Selberg trace formula and to the proof in some cases of the Ramanujan–Petersson conjecture and the global Langlands correspondence for function fields.

As in the first volume we fix a function field $F$ together with a place $\infty$ of $F$ and a positive integer $d$.

The group $F_\infty \backslash GL_d(\mathbb{A})$ acts by right translation on the Hilbert space

$$L^2_{GL_d,1_\infty} = L^2(F_\infty \backslash GL_d(F) \backslash GL_d(\mathbb{A}))$$

and any locally constant function $f$ with compact support on $F_\infty \backslash GL_d(\mathbb{A})$ induces by convolution an operator $R_{GL_d,1_\infty}(f)$ on $L^2_{GL_d,1_\infty}$. This operator admits a kernel

$$K(h,g) = \sum_{\gamma \in GL_d(F)} f(h^{-1}\gamma g)$$

and, at least formally, its trace is the integral

$$J(f) = \int_{F_\infty \backslash GL_d(F) \backslash GL_d(\mathbb{A})} K(g,g) \frac{dg}{d_{\infty}^2}.$$ 

Unfortunately, for an arbitrary function $f$ the operator $R_{GL_d,1_\infty}(f)$ is not of trace class and the integral $J(f)$ is not absolutely convergent. To tide over this difficulty Arthur has introduced a truncated version $J^T(f)$ of the above integral which is absolutely convergent. It depends on some truncation parameter $T$ in the positive Weyl chamber $a^+_w$.

Let us fix some level $I$ and some place $\rho$ which is prime to $I$. If $f = f_\infty f_{\rho_0} f_\rho$, where $f_\infty$ is the very cuspidal Euler–Poincaré function introduced in chapter 5 of the first volume, $f_{\rho_0}$ is an arbitrary element of the Hecke algebra of level $I$ and $f_\rho$ is the Drinfeld function of level $\rho$ (for some positive integer $\rho$) introduced in chapter 4 of the first volume, we will see that $J(f)$ is convergent and that we have

$$J^T(f) = J(f)$$

for any value of the truncation parameter $T$ which is far enough from the walls of $a^+_w$. Moreover, if we take $T$ large enough with respect to $f_{\rho_0}$ (Kazhdan’s trick) we will see that $J(f)$ is equal to the number $\text{Le}_T(f_{\rho_0})$ of fixed points.
The second volume of *Drinfeld modular varieties* is devoted to the Arthur-Selberg trace formula and to the proof in some cases of the Ramanujan-Petersson conjecture and the global Langlands correspondence for function fields.

As in the first volume we fix a function field \( F \) together with a place \( \infty \) of \( F \) and a positive integer \( d \).

The group \( F_{\infty}^* \backslash \GL_d(\mathbb{A}) \) acts by right translation on the Hilbert space

\[
L^2 \left( \GL_d, \chi \right) = L^2 \left( F_{\infty}^* \backslash \GL_d(F)^{\chi} \right)
\]

and any locally constant function \( f \) with compact support on \( F_{\infty}^* \backslash \GL_d(\mathbb{A}) \) induces by convolution an operator \( R_{\GL_d, \chi}(f) \) on \( L^2 \left( \GL_d, \chi \right) \). This operator admits a kernel

\[
K(h, g) = \sum_{\gamma \in \GL_d(F)} f(\gamma^{-1} h g)
\]

and, at least formally, its trace is the integral

\[
J(f) = \int_{F_{\infty}^* \backslash \GL_d(F)^{\chi} \backslash \GL_d(\mathbb{A})} \frac{dg}{d_{\infty} d_g} K(g, g)
\]

Unfortunately, for an arbitrary function \( f \) the operator \( R_{\GL_d, \chi}(f) \) is not of trace class and the integral \( J(f) \) is not absolutely convergent. To tide over this difficulty Arthur has introduced a truncated version \( J^T(f) \) of the above integral which is absolutely convergent. It depends on some truncation parameter \( T \) in the positive Weyl chamber \( \mathfrak{a}_T^+ \).

Let us fix some level \( I \) and some place \( o \) which is prime to \( I \). If \( f = f_\infty f^{(\infty, o)} f_o \), where \( f_\infty \) is the very cuspidal Euler–Poincaré function introduced in chapter 5 of the first volume, \( f^{(\infty, o)} \) is an arbitrary element of the Hecke algebra of level \( I \) and \( f_o \) is the Drinfeld function of level \( r \) (for some positive integer \( r \)) introduced in chapter 4 of the first volume, we will see that \( J(f) \) is convergent and that we have

\[
J^T(f) = J(f)
\]

for any value of the truncation parameter \( T \) which is far enough from the walls of \( \mathfrak{a}_T^+ \). Moreover, if we take \( r \) large enough with respect to \( f^{(\infty, o)} \) (Kazhdan’s trick) we will see that \( J(f) \) is equal to the number \( \text{Lef}_r(f^{(\infty, o)}) \) of fixed points.
of \( \text{Pro}_b^s \times F^{\infty, o} \) acting on the reduction in “characteristic” \( o \) of the Drinfeld modular variety \( M^d_f \).

But, on the one hand, if \( r \) is large enough with respect to \( f^{\infty, o} \) we will see that the Lefschetz number \( \text{Len}_s(f^{\infty, o}) \) is equal to the trace of \( \text{Pro}_b^s \times F^{\infty, o} \) acting on the \( \ell \)-adic cohomology with compact supports of \( \overline{F} \otimes_F M^d_f \). This is due to Grothendieck’s theorem if \( f^{\infty, o} \) is the trivial Hecke operator and it is due to Deligne’s conjecture, proved by Fujiwara and Pink, for a general \( f^{\infty, o} \).

On the other hand, following Arthur the truncated integral \( J^T(f) \) admits a spectral expression and, using some residue computations, which have been done by Waldspurger and which are included as an appendix at the end of this volume, we will make explicit this spectral expression.

Putting together all these results we will obtain an explicit expression for the alternating sum of traces,

\[ \sum_n (-1)^n \text{tr}(\text{Pro}_b^s \times F^{\infty, o}, H^n_{\ell}(\overline{F} \otimes_F M^d_f, \mathbb{Q}_l)), \]

in terms of cuspidal automorphic representations of \( F_s^\infty \setminus GL_{d'}(A) \) (\( d' = 1, \ldots, d \)).

Finally, by a standard procedure we will deduce the Petersson conjecture and the Langlands correspondence for cuspidal automorphic representations of \( F_s^\infty \setminus GL_d(A) \) the local component at \( \infty \) of which is isomorphic to the Steinberg representation.

The numbering of this volume is the continuation of the numbering of the first one. Here is a brief description of its contents.

In chapter 9 we review some basic definitions and results about the cuspidal spectrum of \( L^2_{\text{GL}_d(A)} \).

In chapter 10 we study the geometric side of Arthur’s non-invariant trace formula for our function \( f = f_\infty f^{\infty, o} f_0 \) and we prove that, if \( r \) is large enough with respect to \( f^{\infty, o} \) (and \( f_\infty \)), it has a simple form. In fact \( f_\infty \) may be any very cuspidal function and, as a special case, we obtain the Flicker–Kazhdan simple trace formula. The arguments are adapted from those used by Arthur in the number field case.

In chapter 11 we study the spectral side. Again we adapt Arthur’s arguments. But here we have not been courageous enough to transpose all of his arguments to the function field case. Actually \( J^T(f) \) is a sum over the cuspidal data of expressions \( J^T_c(f) \). Using Waldspurger’s residue computations we obtain an explicit formula for \( J^T_c(f) \) when \( c \) is a regular cuspidal datum. For the other cuspidal data we only state a conjectural formula. This formula has been recently proved by Lafforgue.

In chapter 12 we deduce the Ramanujan–Petersson conjecture and the global Langlands correspondence (for cuspidal automorphic representations of \( F_s^\infty \setminus GL_d(A) \) the local component at \( \infty \) of which is isomorphic to the Steinberg representation) from the results of the previous chapters. We also give a complete description of the virtual \( (\text{Gal}(\overline{F}/F) \times GL_d(A^{\infty})) \)-module

\[ \sum_n (-1)^n H^n_{\ell}(\overline{F} \otimes_F M^d_f, \mathbb{Q}_l). \]

Up to this point we have only considered the cohomology with compact supports of Drinfeld modular varieties. We may also consider the intersection cohomology of the Satake compactification of \( M^d_f \). In chapter 13 we give a conjectural description of the intersection complex of this Satake compactification. We have discovered this conjectural description by transposing to the function field case a formula for the \( L^2 \)-Lefschetz number of a Hecke operator which has been proved by Arthur in the number field case.

There are four appendices. An addendum to appendix D contains some rationality results and a definition of the Grothendieck group of admissible representations.

The main results of reduction theory are reviewed in appendix E. Our proofs differ from Harder’s original ones in the way that we systematically use the Harder–Narasimhan filtration.

In appendix F we give the proof of Harish-Chandra’s results on orbital integrals which are needed in chapter 10.

In appendix G we present some of the basic results concerning the spectral decomposition of Langlands and Morris. In particular we explain the first step in Langlands’ computation of the scalar product of two pseudo-Eisenstein series associated with cuspidal automorphic forms of Levi subgroups.

I would like to thank J.-L. Waldspurger once more for his help during the elaboration of this project. His residue computations are fundamental for the results of the second volume. I would also like to thank R. Pink for his comments on chapter 13. During the preparation of the manuscript I visited the University of Toronto (Winter 1993). My thanks go to J. Arthur for his kind hospitality and for the numerous discussions that I had with him. Special thanks go to the editors who again did a beautiful job for this second volume.
Trace of $f_A$ on the discrete spectrum

(9.0) **Introduction**

In this chapter we will use again the notations of chapters 1, 2, 3 and 6 of the first volume. So $F$ is a function field of positive characteristic $p$, $\infty$ and $o$ are two distinct places of $F$, $I$ is a proper, non-zero ideal of the ring

$$A = \{ a \in F \mid x(a) \geq 0, \forall x \in |X| - \{\infty\} \}$$

such that $o \not\in V(I)$ and $A$ is the ring of adeles of $F$. In fact, in this volume we will use the notation $I$ for the ideal $I$ in order to avoid any confusion with the subsets $I$ of $\Delta$.

The purpose of this chapter is to compute the trace of the compactly supported, locally constant function

$$f_h = f_\infty f_\infty^o f_o$$

acting on $L^2$-automorphic irreducible representations of $F_\infty^x \backslash GL_4(A)$. Here

$$f_\infty \in C_c^\infty(F_\infty^x \backslash GL_4(F_\infty) \backslash \mathcal{E}_\infty)$$

is our very cuspidal Euler–Poincaré function (see (5.2.1)),

$$f_\infty^o \in C_c^\infty(GL_4(A_\infty^o) \backslash K_2^\infty)$$

is an arbitrary Hecke operator and

$$f_o \in C_c^\infty(GL_4(F_o) \backslash K_o)$$
is the Drinfeld function with Satake transform
\[ f^\varphi(z) = p^{\deg(\varphi)(d-1)/2}(z_1^r + \cdots + z_d^r) \]
for some fixed positive integer \( r \).

(9.1) **Automorphic representations**

Let \( M = M_I \) for some \( I \subset \Delta \) and, if \( d_I = (d_1, \ldots, d_s) \) is the corresponding partition of \( d \), \( M \) is canonically isomorphic to \( GL_{d_1} \times \cdots \times GL_{d_s} \). We denote by
\[
C^\infty_M = C^\infty(M(F) \backslash M(A), \mathbb{C})
\]
the \( \mathbb{C} \)-vector space of the complex functions \( \varphi \) on \( M(A) \) which are invariant under left translation by \( M(F) \) and which are invariant under right translation by some compact open subgroup of \( M(A) \) (depending on \( \varphi \)). The unitmodular, locally compact, totally discontinuous, separated topological group \( M(A) \) acts by right translation on \( M(F) \backslash M(A) \) and therefore acts smoothly on \( C^\infty_M \). We denote this action by \( R_M \).

An **automorphic form** for \( M \) is a function \( \varphi \in C^\infty_M \) such that the subrepresentation of \( (C^\infty_M, R_M) \) generated by \( \varphi \), i.e. the \( \mathbb{C} \)-linear span of
\[
R_M(M(A))(\varphi) \subset C^\infty_M,
\]
is admissible. We denote by
\[
A_M = A(M(F) \backslash M(A), \mathbb{C}) \subset C^\infty_M
\]
the subset of the automorphic forms for \( M \). Obviously, \( A_M \) is a \( \mathbb{C} \)-vector subspace and is stable under \( R_M(M(A)) \) (the subquotients and the finite sums of admissible representations are admissible). We again denote by \( R_M \) the restriction of \( R_M \) to \( A_M ; (A_M, R_M) \) is a smooth representation of \( M(A) \).

An **automorphic irreducible representation** of \( M(A) \) is an irreducible representation of \( M(A) \) which is isomorphic to a subquotient of \( (A_M, R_M) \).

**Lemma (9.1.3).** — Any automorphic irreducible representation of \( M(A) \) is admissible and is isomorphic to
\[
(W_2, \rho_2)/(W_1, \rho_1)
\]
where
\[
(W_1, \rho_1) \subsetneq (W_2, \rho_2)
\]
are admissible subrepresentations of \( (A_M, R_M) \).

**Proof:** Up to isomorphism the automorphic irreducible representation \( \varphi \times \pi \) of \( (M, \rho) \) is equal to
\[
(W_2, \rho_2)/(W_1, \rho_1)
\]
for some subrepresentations
\[
(W_1, \rho_1) \subseteq (W_2, \rho_2) \subset (A_M, R_M).
\]
Let us arbitrarily fix \( \varphi \in W_2 - W_1 \). As \( \varphi \times \pi \) is irreducible the subrepresentation \( (W_2, \rho_2) \) generated by \( \varphi \) maps onto \( \varphi \times \pi \). Therefore we may assume that \( (W_2, \rho_2) \) is generated by \( \varphi \). Then, as \( \varphi \) is an automorphic form, \( (W_2, \rho_2) \) is admissible and the lemma follows (see (D.2)).

Let \( Z_M \) be the center of \( M \). If we identify \( M \) with \( GL_{d_1} \times \cdots \times GL_{d_s} \), \( (M = M_I \) and \( d_i = (d_1, \ldots, d_s) \)), \( Z_M \) is identified with \( (GL_1)^s \).

Let us define
\[
\deg : A^\times \longrightarrow \mathbb{Z}
\]
by
\[
\deg(a) = \frac{1}{f} \sum_{x \in |X|} \deg(x)(a) \quad (\forall a \in A^\times),
\]
where \( f \) is the degree over \( \mathbb{F}_p \) of the field of constants in \( F \) (for each \( x \in |X| \), \( f \) divides \( \deg(x) \)). Then \( \deg \) is a group homomorphism and is trivial on \( F^\times \subset A^\times \) and \( \mathbb{O}^\times \subset A^\times \). If we denote by \( (A^\times)^1 \) the kernel of \( \deg \),
\[
F^\times/(A^\times)^1 \cong \text{Pic}_{X/\mathbb{F}_p}(\mathbb{F}_p)
\]
is finite and \( F^\times/(A^\times)^1 \) is compact and we have an exact sequence of abelian groups,
\[
1 \longrightarrow F^\times/(A^\times)^1 \longrightarrow F^\times \longrightarrow \mathbb{Z} \quad \text{deg}.
\]
Moreover we have

**Lemma (9.1.4).** — The homomorphism \( \deg : F^\times \longrightarrow \mathbb{Z} \) is onto.

**Proof:** The zeta function of \( X \) is a rational function (see [We 4] Ch. 4, 22). More precisely we have
\[
\prod_x (1 - T^{\deg(x)})^{-1} = \frac{P(T)}{(1 - T)(1 - p^iT)}
\]
for some polynomial \( P(T) \in \mathbb{Q}[T] \). Therefore, if \( e \) is the greatest common divisor of the integers \( \deg(x) \) (\( x \in |X| \)) the rational function
\[
\frac{P(T)}{(1 - T)(1 - p^iT)}
\]
is invariant under \( T \mapsto \zeta T \) for any \( e \)-th root of unity \( \zeta \). But, by the Riemann hypothesis for curves (see loc. cit.), \( P(T) \) is prime to \( (1 - T)(1 - p^iT) \). Thus we should have \( e = 1 \).
Therefore, if we define
\[ \text{deg}_{Z_M} : Z_M(A) \to \mathbb{Z}^s \]
by
\[ \text{deg}_{Z_M}(z_{A,1}, \ldots, z_{A,s}) = (\deg(z_{A,1}), \ldots, \deg(z_{A,s})) \]
and if we set
\[ Z_M(A)^1 = \text{Ker}(\text{deg}_{Z_M}) \]
we have \( Z_M(F) \subseteq Z_M(A)^1 \) and \( Z_M(O) \subseteq Z_M(A)^1 \). The group \( Z_M(F) \setminus Z_M(A)^1 / Z_M(F) \) is finite and the group \( Z_M(F) \setminus Z_M(A)^1 \) is compact. We have an exact sequence of abelian groups,
\[ 1 \to Z_M(F) \setminus Z_M(A)^1 \to Z_M(F) \setminus Z_M(A) \xrightarrow{\text{deg}_{Z_M}} \mathbb{Z}^s \to 0. \]

Let \((\mathcal{V}, \pi)\) be a smooth representation of the locally compact, totally disconnected, separated, abelian group \( Z_M(F) \setminus Z_M(A) \). A vector \( v \in \mathcal{V} \) is said to be \( Z_M(A) \)-finite if the \( C \)-linear span of \( \pi(Z_M(F) \setminus Z_M(A))(v) \) in \( \mathcal{V} \) is finite dimensional.

We denote by \( \mathcal{X}_M \) the abelian group of the smooth complex characters of \( Z_M(F) \setminus Z_M(A) \).

**Lemma (9.1.5).** — Let \( v \in \mathcal{V} \). Then the following conditions are equivalent:

(i) \( v \) is \( Z_M(A) \)-finite,

(ii) there exists a function \( \mu : \mathcal{X}_M \to \mathbb{Z}_{\geq 0} \) with finite support (i.e., with \( \mu(\chi) = 0 \) for almost all \( \chi \in \mathcal{X}_M \)) such that
\[ \left( \prod_{\chi \in \mathcal{X}_M} \mu(\chi) \prod_{i=1}^{\mu(\chi)} (\pi(z_{A,1}, \ldots, z_{A,i})(v) - \chi(z_{A,i})) \right)(v) = 0 \]
for every family \( (z_{A,1}, \ldots, z_{A,i}) \) of elements of \( Z_M(A) \),

(iii) there exist finitely many characters \( \chi_1, \ldots, \chi_N \) in \( \mathcal{X}_M \) and a positive integer \( m \) such that
\[ \left( \prod_{n=1}^{N} (\pi(z_{A}) - \chi_n(z_{A}))^m \right)(v) = 0 \]
for every \( z_{A} \in Z_M(A) \).

\[ \text{Proof:} \] Let \( K' \subseteq Z_M(A) \) be a compact open subgroup such that \( v \in \mathcal{V}^{K'} \) and let
\[ Z = Z_M(F) \setminus Z_M(A)/K'. \]
The abelian group \( Z \), and hence its group algebra \( C[Z] \), acts on \( \mathcal{V} \). Let \( \mathcal{J} \) be the annihilator of \( v \) in \( C[Z] \). Condition (i) is equivalent to
\[ \dim_{C}(C[Z]/\mathcal{J}) < +\infty. \]

If it is satisfied there are finitely many maximal ideals \( p_1, \ldots, p_N \) in \( C[Z] \) such that
\[ \sqrt{\mathcal{J}} = p_1 \cap \cdots \cap p_N = p_1 \cdots p_N \]
and a positive integer \( m \) such that
\[ p_1^m \cdots p_N^m = (\sqrt{\mathcal{J}})^m \subseteq \mathcal{J}, \]
i.e. \( (\sqrt{\mathcal{J}}/\mathcal{J})^m = (0) \) in \( C[Z]/\mathcal{J} \). For each \( n = 1, \ldots, N \) there exists a unique \( \chi_n \in \mathcal{X}_M \) such that
\[ \ker(\chi_n) = p_n \]
(we also denote by \( \chi_n \) the \( C \)-linear extension of \( \chi_n \) to \( C[Z] \)). Therefore condition (ii) is satisfied if we define \( \mu(\chi_n) = m \) for \( n = 1, \ldots, N \) and by \( \mu(\chi) = 0 \) for \( \chi \in \mathcal{X}_M - \{ \chi_1, \ldots, \chi_N \} \).

Obviously condition (ii) implies condition (iii).

Finally, if condition (iii) is satisfied we may assume that \( \chi_1, \ldots, \chi_N \) are trivial on \( K' \subseteq Z_M(A) \). The abelian group \( Z \) is finitely generated as it is an extension of \( Z' \) by the finite group \( Z_M(F) \setminus Z_M(A)^1/K' \). Therefore the ideal \( \mathcal{I} \) in \( C[Z] \) generated by the elements
\[ \prod_{n=1}^{N} (z - \chi_n(z))^m \quad (z \in \mathcal{Z}) \]
has finite codimension in \( C[Z] \) if \( \{z_1, \ldots, z_r\} \) generates the abelian group \( Z \) then
\[ \{z_1^\alpha \cdots z_r^\alpha \mid \alpha \in \{0, 1, \ldots, Nm - 1\}^r\} \]
generates the \( C \)-vector space \( C[Z]/\mathcal{I} \). But \( \mathcal{I} \subseteq \mathcal{J} \) and therefore condition (i) is satisfied.

For each \( \chi \in \mathcal{X}_M \) we set
\[ \mathcal{V}_{\chi, \text{gen}} = \bigcup_{m \geq 1} \mathcal{V}_{\chi, m} \]
where, for each positive integer \( m \), we have put
\[ \mathcal{V}_{\chi, m} = \{ v \in \mathcal{V} \mid (\pi(z_{A}) - \chi(z_{A}))^m(v) = 0, \forall z_{A} \in Z_M(A) \}. \]
Let
\[ V_f \subset V \]
be the subset of all \( Z_M(A) \)-finite vectors in \( V \). Then \( V_{X,\text{gen}} \), \( V_{X,\text{gen}} \) and \( V_f \) are \( \mathbb{C} \)-vector subspaces of \( V \) and are stable under \( \pi(Z_M(F) \backslash Z_M(A)) \). We have
\[ \sum_{X \in X_M} V_{X,\text{gen}} \subset V_f \]
(see (9.1.5)).

**Corollary (9.1.6).** — The \( \mathbb{C} \)-vector space \( V_f \) is the direct sum of its subspaces \( V_{X,\text{gen}} \) (\( X \in X_M \)).

**Proof:** Let \( \chi', \chi'' \in X_M \) with \( \chi' \neq \chi'' \), let \( m', m'' \) be positive integers and let \( v \in V_f \). We assume that
\[ (\pi(z_A) - \chi'(z_A))^{m'}(v) = 0 = (\pi(z_A) - \chi''(z_A))^{m''}(v) \]
for every \( z_A \in Z_M(A) \). Let us fix \( z_A \in Z_M(A) \) such that \( \chi'(z_A) \neq \chi''(z_A) \). Then by the Bezout theorem there exist \( P'(T), P''(T) \in \mathbb{C}[T] \) such that
\[ P'(T)(T - \chi'(z_A))^{m'} + P''(T)(T - \chi''(z_A))^{m''} = 1. \]
Replacing \( T \) by \( \pi(z_A) \) and applying the operator to \( v \) we get \( 0 = v \). Therefore we have proved that \( V_{X,\text{gen}} \cap V_{X',\text{gen}} = \{0\} \).

Let \( v \in V_f \). Thanks to (9.1.5) there exists \( \mu : X_M \to \mathbb{Z}_{\geq 0} \) such that
\[ \left( \prod_{X \in X_M} (\pi(z_{A,X}) - \chi(z_{A,X}))^{\mu(X)} \right)(v) = 0 \]
for every family \( (z_{A,X})_{X \in X_M} \in (Z_M(A))^{X_M} \). Let us prove that
\[ v \in \sum_{X \in X_M} V_{X,\text{gen}} \]
by induction on the number of elements in the support of \( \mu \).

If \( \text{Supp}(\mu) = \{X\} \) we have \( v \in V_{X,\text{gen}} \). If \( \chi' \neq \chi'' \) are in \( \text{Supp}(\mu) \) let us fix \( z_A \in Z_M(A) \) such that \( \chi'(z_A) \neq \chi''(z_A) \). Then by the Bezout theorem there exist polynomials \( P'(T), P''(T) \in \mathbb{C}[T] \) such that
\[ P'(T)(T - \chi'(z_A))^{\mu(X')} + P''(T)(T - \chi''(z_A))^{\mu(X'')} = 1. \]
Let us set
\[ v' = (P'(\pi(z_A))(\pi(z_A) - \chi'(z_A))^{\mu(X')})(v) \]
and
\[ v'' = (P''(\pi(z_A))(\pi(z_A) - \chi''(z_A))^{\mu(X'')})(v). \]
We have
\[ v = v' + v'' \]
and
\[ \left( \prod_{X \in X_M \atop X \neq X'} (\pi(z_{A,X}) - \chi(z_{A,X}))^{\mu(X)} \right)(v') = 0 \]
and
\[ \left( \prod_{X \in X_M \atop X \neq X''} (\pi(z_{A,X}) - \chi(z_{A,X}))^{\mu(X)} \right)(v'') = 0 \]
for every family \( (z_{A,X})_{X \in X_M} \in Z_M(A)^{X_M} \). By the induction hypothesis we have
\[ v', v'' \in \sum_{X \in X_M} V_{X,\text{gen}} \]
and the corollary follows. \( \square \)

Let
\[ C_{Z_M}^{\infty} = C^{\infty}(Z_M(F) \backslash Z_M(A), C) \]
be the \( \mathbb{C} \)-vector space of the complex functions \( \varphi \) on \( Z_M(A) \) which are invariant under \( Z_M(F) \) and under some compact open subgroup of \( Z_M(A) \) (depending on \( \varphi \)). The group \( Z_M(F) \backslash Z_M(A) \) acts smoothly by translation on \( C_{Z_M}^{\infty} \).

For each \( \chi \in X_M \) let
\[ (9.1.7) \quad \iota_{\chi} : C[X_1, \ldots, X_s] \longrightarrow C_{Z_M}^{\infty} \]
be the \( \mathbb{C} \)-linear map defined by
\[ \iota_{\chi}(P(X))(z_A) = P(\text{deg}_{Z_M}(z_A)) \chi(z_A) \]
for every \( P(X) \in C[X_1, \ldots, X_s] \) and every \( z_A \in Z_M(A) \).

We denote by \( C[X_1, \ldots, X_s]_m \) the \( \mathbb{C} \)-vector space of the polynomials \( P(X) \) in \( C[X_1, \ldots, X_s] \) such that the degree of \( P(X) \) in each variable is strictly smaller than \( m \).

**Lemma (9.1.8).** — The map \( \iota_{\chi} \) is injective and its image is exactly the subspace \( (C_{Z_M}^{\infty})_{X,\text{gen}} \) of \( C_{Z_M}^{\infty} \). More precisely, for each positive integer \( m \) we have
\[ \iota_{\chi}(C[X_1, \ldots, X_s]_m) = (C_{Z_M}^{\infty})_{X,m}. \]
9. TRACE OF $f_A$ ON THE DISCRETE SPECTRUM

Let us define a homomorphism

$$\deg_M : M(A) \longrightarrow \bigoplus_{j=1}^{n} \frac{1}{d_j} \mathbb{Z} \subset \mathbb{Q}^*$$

by

$$\deg_M(m_A) = \left( \frac{\deg(\det(g_{a1}A \cdots, g_{an}A))}{d_1}, \ldots, \frac{\deg(\det(g_{a1}A \cdots, g_{an}A))}{d_n} \right)$$

for all $m_A = (g_{a1}A \cdots, g_{an}A) \in M(A) \cong GL_{d_1}(A) \times \cdots \times GL_{d_n}(A)$. We have

$$\deg_M|Z_M(A) = \deg_{Z_M}.$$ 

Let $M(A)^1 \subset M(A)$ be the kernel of $\deg_M$. We have $M(F) \subset M(A)^1$ and $M(O) \subset M(A)^1$ and we have

$$Z_M(A)^1 = Z_M(A) \cap M(A)^1.$$ 

For each $\chi \in \mathcal{X}_M$ let us denote by

$$(9.1.9) \quad C_{M,M}^{\chi} = C_{\infty}^{\chi}(M(F) \backslash M(A), C) \subset C_{\infty}^{\chi}$$

the C-vector subspace of the functions $\varphi \in C_{\infty}^{\chi}$ such that

$$\varphi(z m_A) = \chi(z) \varphi(m_A) \quad (\forall z \in Z_M(A), m_A \in M(A)).$$

The subspace $C_{M,M}^{\chi}$ is stable under $R_M(M(A))$ and we will denote by $R_{M,M}^{\chi}$ the restriction of $R_M$ to $C_{M,M}^{\chi}$.

We consider the C-linear map

$$(9.1.10) \quad \iota : \bigoplus_{\chi \in \mathcal{X}_M} C[X_1, \ldots, X_s] \otimes C_{M,M}^{\chi} \longrightarrow C_{M,M}^{\chi}$$

defined by

$$\iota(P(X) \otimes \varphi)(m_A) = P(\deg_M(m_A)) \varphi(m_A)$$

for every $P(X) \in \mathbb{C}[X_1, \ldots, X_s]$, every $\varphi \in C_{M,M}^{\chi}$ and every $\chi \in \mathcal{X}_M$. On $C[X_1, \ldots, X_s]$ we have the action $r$ of $\mathbb{Q}^s$ defined by

$$r(x)(P(X)) = P(X + x)$$

$(\forall x \in \mathbb{Q}^s, \forall P(X) \in \mathbb{C}[X_1, \ldots, X_s])$. Therefore $M(A)$ acts on $C[X_1, \ldots, X_s]$ by $r \circ \deg_M$ and on the source of $\iota$ by

$$\bigoplus_{\chi \in \mathcal{X}_M} (r \circ \deg_M) \otimes R_{M,M}.$$ 

For this action on the source of $\iota$ and for the action $R_M$ of $M(A)$ on its target $\iota$ is obviously $M(A)$-equivariant.
9. Trace of $f_A$ on the discrete spectrum

is invertible and if we denote by $(c_{\beta,n})$ the inverse of this matrix, we have

$$
\sum_{\alpha} (\alpha) \varphi_\alpha(m_A) \deg_M(m_A)^{\alpha - \beta} = \sum_{n=1}^{m*} c_{\beta,n} \varphi(z_{A,n}m_A)
$$

for every $m_A \in M(A)$ and every $\beta \in \{0, 1, \ldots, m-1\}^s$, where $\alpha$ runs through the set $\prod_{j=1}^{s} \{\beta_j, \beta_j + 1, \ldots, m - 1\}$. But, if we set

$$
\psi_\beta = \sum_{\alpha} (\alpha) \varphi_\alpha(\deg_M)^{\alpha - \beta},
$$

we have

$$
\varphi_\alpha = \sum_{\beta} (-1)^{\beta - \alpha} \left(\frac{\beta}{\alpha}\right) \psi_\beta \deg_M^{\beta - \alpha}
$$

where $\beta$ runs through the set $\prod_{j=1}^{s} \{\alpha_j, \alpha_j + 1, \ldots, m - 1\}$ and where $|\beta - \alpha| = (\beta_1 - \alpha_1) + \cdots + (\beta_s - \alpha_s)$. Therefore we obtain

$$
\varphi_\alpha(m_A) = \sum_{\beta} (-1)^{\beta - \alpha} \left(\frac{\beta}{\alpha}\right) \sum_{n=1}^{m*} c_{\beta,n} \varphi(z_{A,n}m_A) \deg_M(m_A)^{\beta - \alpha}.
$$

\textbf{Lemma (9.1.13).} — Let $(\mathcal{V}, \pi)$ be an admissible representation of $Z_M(F) \setminus M(A)$. Then we have

$$\mathcal{V} = \mathcal{V}^f$$

(any vector in $\mathcal{V}$ is $Z_M(A)$-finite).

Moreover, if $(\mathcal{V}, \pi)$ is irreducible there exists a unique $\chi \in X_M$ such that

$$\mathcal{V} = \mathcal{V}^\chi$$

(existence of a central character).

\textbf{Proof:} Let $v \in \mathcal{V}$ and let $K'$ be a compact open subgroup of $M(A)$ such that $v \in \mathcal{V}^{K'}$. Then $\pi(Z_M(A))(v)$ is contained in $\mathcal{V}^{K'}$ and $\dim_{\mathbb{C}}(\mathcal{V}^{K'}) < +\infty$. Therefore $v$ is $Z_M(A)$-finite.

If $(\mathcal{V}, \pi)$ is irreducible we have $\mathcal{V} = \mathcal{V}_{X,m}$ for any $\chi \in X_M$ and any positive integer $m$ such that $\mathcal{V}_{X,m} \neq 0$ ($\mathcal{V}_{X,m}$ is stable under $\pi(M(A))$). Therefore there exists a unique $\chi \in X_M$ such that $\mathcal{V}_{X,m} \neq 0$ for some $m$. Let $m$ be the smallest positive integer such that $\mathcal{V}_{X,m} \neq 0$. For any $z_A \in Z_M(A)$ we have $(\pi(z_A) - \chi(z_A))\mathcal{V}_{X,m} \subset \mathcal{V}_{X,m-1} = 0$ and therefore $m = 1$. See also (D.1.12).
9. Trace of \( \mathcal{I}_A \) on the Discrete Spectrum

**Corollary (9.1.16).** — An irreducible representation of \( M(A) \) is automorphic if and only if it is isomorphic to a subquotient of \( (\mathcal{A}_{M,\chi}, R_{M,\chi}) \) for some \( \chi \in \mathcal{X}_M \) (this \( \chi \), if it exists, is the central character of the representation and is thus uniquely determined).

Here we again denote by \( R_{M,\chi} \) the restriction of \( R_{M,\chi} \) to \( \mathcal{A}_{M,\chi} \subset \mathcal{C}^\infty_{M,\chi} \).

**Proof:** The "if" part is obvious. Conversely, let \( (\mathcal{V}, \pi) \) be an automorphic irreducible representation of \( M(A) \). It is admissible (see (9.1.13)) and trivial on \( Z_M(F) \). Therefore we have \( \mathcal{V} = \mathcal{V}_{\chi, 1} \) for some \( \chi \in \mathcal{X}_M \) (see (9.1.13)).

Let us choose admissible subrepresentations

\[
(\mathcal{W}_1, \rho_1) \subseteq (\mathcal{W}_2, \rho_2) \subset (\mathcal{A}_M, R_M)
\]

such that \( (\mathcal{V}, \pi) \) is isomorphic to

\[
(\mathcal{W}_2, \rho_2)/(\mathcal{W}_1, \rho_1)
\]

(see (9.1.3)). Replacing \( \mathcal{W}_1 \) and \( \mathcal{W}_2 \) by \( (\mathcal{W}_1)_{\chi, m} \) and \( (\mathcal{W}_2)_{\chi, m} \) respectively for some sufficiently large enough positive integer \( m \) we may assume that

\[
\mathcal{W}_1 \subseteq \mathcal{W}_2 \subset (\mathcal{A}_M)_{\chi, m}
\]

for some \( m \) (see (9.1.15); choose \( m \) such that the image of \( (\mathcal{W}_2)_{\chi, m} \) in \( \mathcal{V} \) is non-zero).

Let \( \mathcal{K} \) be a compact open subgroup of \( M(A) \) such that \( \mathcal{V}^{\mathcal{K}} \neq (0) \). Then we have \( \mathcal{W}_2^{\mathcal{K}} \neq (0) \) (the functor \( (\cdot)^{\mathcal{K}} \) is exact, see (D.1.5)). Let \( (\mathcal{W}_2, \rho_2') \) be a subrepresentation of \( (\mathcal{W}_2, \rho_2) \) such that \( \mathcal{W}_2^{\mathcal{K}} \) maps onto \( \mathcal{V}^{\mathcal{K}} \) and has the smallest possible dimension for this property. Let \( (\mathcal{W}_2', \rho_2') \) be the intersection of all the subrepresentations of \( (\mathcal{W}_2, \rho_2) \) containing \( \mathcal{W}_2^{\mathcal{K}} \). Then if \( (\mathcal{W}_2', \rho_2') \) is a proper subrepresentation of \( (\mathcal{W}_2', \rho_2') \) the map \( (\mathcal{W}_2', \rho_2') \to (\mathcal{V}, \pi) \) is zero (otherwise it would be surjective and \( (\mathcal{W}_2')^{\mathcal{K}} \to \mathcal{V}^{\mathcal{K}} \) would be surjective too, so that the inclusion

\[
(\mathcal{W}_2', \rho_2')^{\mathcal{K}} \subset (\mathcal{W}_2')^{\mathcal{K}} = (\mathcal{W}_2')^{\mathcal{K}}
\]

would be an equality and we would have a contradiction). Therefore, replacing \( (\mathcal{W}_2, \rho_2) \) by \( (\mathcal{W}_2', \rho_2') \) and \( (\mathcal{W}_1, \rho_1) \) by its intersection with \( (\mathcal{W}_2', \rho_2') \) we may assume that any proper subrepresentation of \( (\mathcal{W}_2', \rho_2') \) is contained in \( (\mathcal{W}_1, \rho_1) \).

Now to prove the corollary it is sufficient to construct a non-zero homomorphism \( (\mathcal{W}_2, \rho_2) \to (\mathcal{A}_{M, \chi}, R_{M, \chi}) \). But applying proposition (9.1.15) we obtain a non-zero homomorphism

\[
(\mathcal{W}_2, \rho_2) \to \mathcal{C}[X_1, \ldots, X_s] \otimes \mathcal{C} \mathcal{A}_{M, \chi}(r \circ \det_M) \otimes R_{M, \chi}
\]
and we leave it to the reader to check that there is at least one filtration

\[(0) = (U_0, \sigma_0) \subset (U_1, \sigma_1) \subset \cdots \subset (U_L, \sigma_L) = (C[X_1, \ldots, X_s], m, r \circ \det_M)\]

such that the successive quotients

\[(U_{\ell}, \sigma_\ell)/(U_{\ell-1}, \sigma_{\ell-1}) \quad (\ell = 1, \ldots, L)\]

are isomorphic to the trivial representation \((C, 1)\) of \(M(\mathbb{A})\). By the proof of the corollary is completed. \(\square\)

### 9.2.2 Cuspidal automorphic representations

Let \(M = M_J\) be a standard Levi subgroup of \(GL_d\) as in (9.1). Let \(P' \subset M\) be a standard parabolic subgroup of \(M\) with its standard Levi decomposition \(P' = M' N', P' = \prod_{j \in I} M'_j N'_j = N'_J \cap M_I\) for some \(J \subset I\).

**Lemma (9.2.1).** — The topological space \(N'(F') \backslash N'(\mathbb{A})\) is compact.

**Proof:** Let \(R^+_M\) and \(R^+_M\) be the sets of positive roots for \((M, T, B \cap M)\) and \((M', T, B \cap M')\) and for each \(\beta = \epsilon_i - \epsilon_j \in R^+_M - R^+_M\), let

\[x_\beta : G_a \rightarrow N', \ t \mapsto 1 + tE_{ij},\]

be the corresponding 1-parameter subgroup (\(E_{ij}\) is the elementary matrix with all entries 0 except the entry on the \(i\)-th row and the \(j\)-th column which is equal to 1). Then we have

\[N' = \prod_{\beta \in R^+_M - R^+_M} x_\beta(G_a).\]

There exists a total ordering \(\beta_1 < \beta_2 < \cdots < \beta_L\) on \(R^+_M - R^+_M\) such that, for each \(\ell = 0, 1, \ldots, L\),

\[V_\ell = x_{\beta_1}(G_a) \cdots x_{\beta_\ell}(G_a)\]

is a normal closed algebraic subgroup of \(N'\). As \(V_\ell/V_{\ell-1}\) is isomorphic to \(G_a\) for \(\ell = 1, \ldots, L\) and as \(F\backslash A\) is compact (the group \(F\backslash A/\mathcal{O}_X = H^1(X, \mathcal{O}_X)\) is finite), \(V_\ell(F)\backslash V_\ell(A)\) is compact for \(\ell = 1, \ldots, L\) (induction on \(\ell\)) and the lemma is proved (\(V_L = N'\)). \(\square\)

For any \(\varphi \in C_{c}^\infty(\mathbb{A})\) we set

\[(9.2.2) \quad \varphi_{P'}(m_\mathbb{A}) = \frac{1}{\text{vol}(N'(F') \backslash N'(A), d\nu' \backslash d\nu)} \int_{N'(F') \backslash N'(A)} \varphi(n'_\mathbb{A} m_\mathbb{A}) \frac{dn'_\mathbb{A}}{dn'_{\mathbb{A}}} \quad (\forall m_\mathbb{A} \in M(\mathbb{A}))\]

where \(dn'_\mathbb{A}\) is the Haar measure on \(N'(A)\) which is normalized by \(\text{vol}(N'(F') \cap K_T, d\nu') = 1\) and \(d\nu'\) is the counting measure on \(N'(F)\). The function

\[\varphi_{P'} : M'(F') \backslash N'(A) \backslash M(A) \rightarrow C\]

is called the constant term of \(\varphi\) along \(P'\). It is invariant under right translation by some compact open subgroup of \(M(\mathbb{A})\).

The function \(\varphi \in C_{c}^\infty(\mathbb{A})\) is said to be cuspidal if

\[(9.2.3) \quad \varphi_{P'}(m_\mathbb{A}) = 0 \quad (\forall m_\mathbb{A} \in M(\mathbb{A}))\]

for any proper standard parabolic subgroup \(P'\) of \(M\). We denote by

\[C_{c}^\infty_{M, \text{cusp}} = C_{c}^\infty(M(F) \backslash M(\mathbb{A}), C) \subset C_{c}^\infty(\mathbb{A})\]

the \(C\)-vector subspace of the cuspidal \(\varphi \in C_{c}^\infty(\mathbb{A})\). It is invariant under \(R_M(M(\mathbb{A}))\). We denote by \(R_{M, \text{cusp}}\) the restriction of \(R_M\) to \(C_{c}^\infty_{M, \text{cusp}}\).

A **cusp form** for \(M\) is an automorphic form for \(M\) which is cuspidal. We set

\[(9.2.4) \quad A_{M, \text{cusp}} = A_{\text{cusp}}(M(F) \backslash M(\mathbb{A}), C) = A_M \cap C_{c}^\infty_{M, \text{cusp}}.\]

This is a \(C\)-vector subspace of \(A_M\) which is invariant under \(R_M(M(\mathbb{A}))\). We again denote by \(R_{M, \text{cusp}}\) the restriction of \(R_M\) to \(A_{M, \text{cusp}}\).

**Lemma (9.2.5).** — The isomorphism \(\iota\) of (9.1.11) (resp. (9.1.15)) induces an isomorphism from

\[\bigoplus_{x \in X_M} C[X_1, \ldots, X_s] \otimes_C C_{c}^\infty_{M, X, \text{cusp}}\]

(resp.

\[\bigoplus_{x \in X_M} C[X_1, \ldots, X_s] \otimes_C A_{M, X, \text{cusp}}\]

onto \(C_{c}^\infty(\mathbb{A}) \cap C_{c}^\infty_{M, \text{cusp}}\) (resp. \(A_{M, X, \text{cusp}}\)) where we have set

\[C_{c}^\infty_{M, X, \text{cusp}} = C_{c}^\infty_{M, X} \cap C_{c}^\infty_{M, \text{cusp}}\]

(resp.

\[A_{M, X, \text{cusp}} = A_{M, X} \cap A_{M, \text{cusp}}.\]
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Proof: The lemma follows from remark (9.1.12): for any \( P(X) \in C_0[X_1, \ldots, X_n] \), any \( \psi \in C_0^\infty \) and any standard parabolic subgroup \( P' \) of \( M \), we have
\[
(P(\deg_M)\psi)_{P'} = P(\deg_M)\psi_{P'}.
\]
\(\square\)

**Theorem (9.2.6) (Harder).** Let \( K' \) be a compact open subgroup of \( M(A) \). Then there exists an open subset \( C_{K'} \) in \( M(A) \) such that \( Z_M(A)(F)C_{K'}K' = C_{K'} \), the quotient \( Z_M(A)(F)C_{K'}/K' \) is finite and
\[
\text{Supp}(\varphi) \subset C_{K'} \quad (\forall \varphi \in \langle C_{M,cusp}(K') \rangle).
\]

Before proving the theorem we need to recall some basic results from reduction theory.

Recall that \( M = M_I \). For any integers \( c_1 \leq c_2 \) let
\[
T(A)^{[\infty, c_1]} \quad (\text{resp. } T(A)^{[c_1, c_2]}(\alpha))
\]
be the open subset of \( T(A) \) defined by the conditions
\[
\deg(\alpha(\mathcal{A})) \leq c_2 \quad (\forall \alpha \in I)
\]
(resp.
\[
c_1 \leq \deg(\alpha(\mathcal{A})) \leq c_2 \quad (\forall \alpha \in I).
\]

Let \( g_X \) be the geometric genus of \( X \), i.e. the genus of an arbitrary connected component of \( X \otimes_{F_\mathcal{P}} \bar{k} \) where \( k \) is an algebraic closure of \( \mathbb{F}_\mathcal{P} \). We have
\[
dim_{\mathbb{F}_\mathcal{P}}(H^1(X, \mathcal{O}_X)) = f g_X.
\]

**Lemma (9.2.7) (Harder).** For any integer \( c_2 \geq 2g_X \) we have
\[
M(A) = M(F)U^I(A)T(A)^{[\infty, c_1]}M(O)
\]
(recall that \( U^I = U \cap M_I \)).

A proof of (9.2.7) is given in (E.1.1).

**Lemma (9.2.8) (Harder).** For any compact open subgroup \( K' \) of \( M(A) \) and any integer \( c_2 \) there exists an integer \( c_1 \leq c_2 \) having the following property: if \( \alpha \in I \) and \( N' = N_{I - (\alpha)}^I \) we have
\[
N'(A) = N'(F)(N'(\mathcal{A}) \cap \mathcal{A})K'(b_{\mathcal{A}}^{-1})
\]
for any \( b_{\mathcal{A}} = u_{\mathcal{T}}^I \in B^I(A) = U^I(A)T(A) \) with \( t_{\mathcal{T}} \in T(A)^{[\infty, c_1]} \) and with \( \deg(\alpha(t_{\mathcal{T}})) < c_1 \).
Now, choosing a total ordering $\beta_1 < \beta_2 < \ldots < \beta_L$ on $R^+_M - R^+_M$, as in the proof of (9.2.1), we obtain a tower of normal closed algebraic subgroups of $N'$,

$$ (0) = V_0 \subset V_1 \subset \cdots \subset V_L = N' $$

with

$$ V_\ell / V_{\ell - 1} \cong G_{\alpha} \quad (\forall \ell = 1, \ldots, L) $$

(set $V_\ell = x_{\beta_\ell} G_{\alpha} \cdots x_{\beta_1} G_{\alpha}$). Then by induction on $\ell$ we easily check that

$$ V_\ell(A) = V_\ell(F)(V_\ell(A) \cap b_\alpha^L K'(...(b_\alpha^1)^-1)) $$

for any $\ell = 1, \ldots, L$ and any $b_\alpha^L = u_\alpha^L t_\alpha \in B^L(A) = U^L(A)T(A)$ with $t_\alpha \in T(A)_1^{[-\infty, c_2]}$ and with

$$ \deg(\alpha(t_\alpha)) < c_1. $$

This completes the proof of the lemma.

**Corollary (9.2.9).** — For any compact open subgroup $K'$ of $M(A)$ and any integer $c_2$ there exists an integer $c_1 \leq c_2$ having the following property: if $\alpha \in I$ ($M = M_1$) and if $P' = P_1^{-\{\alpha\}}$ is the corresponding maximal proper standard parabolic subgroup of $M$ we have

$$ \varphi(u_\alpha^L t_\alpha m_{\alpha}) = \varphi_{P'}(u_\alpha^L t_\alpha m_{\alpha}) $$

for any $\varphi \in (C^\infty_M)_{K'}^\alpha$, any $u_\alpha^L \in U_\alpha^L$, any $t_\alpha \in T(A)_1^{[-\infty, c_2]}$ with $\deg(\alpha(t_\alpha)) < c_1$ and any $m_{\alpha} \in M(O)$.

**Proof:** Replacing $K'$ by an open subgroup we may assume that $K'$ is an open normal subgroup of $M(O)$. Let us fix an integer $c_1 \leq c_2$ satisfying the property of (9.2.8) and let $\alpha \in I$. Then we have

$$ N'(A) = N'(F)(N'(A) \cap (b_\alpha^L m_{\alpha} K' m_{\alpha}^{-1}(b_\alpha^L)^{-1})) $$

for any $b_\alpha^L = u_\alpha^L t_\alpha \in U^L(A)T(A)_1^{[-\infty, c_2]}$ with $\deg(\alpha(t_\alpha)) < c_1$ and any $m_{\alpha} \in M(O)$ ($K'$ is normal in $M(O)$). Therefore, for any such $b_\alpha^L$ and $m_{\alpha}$ the function

$$ N'(A) \rightarrow C, \quad n_{\alpha} \mapsto \varphi(n_{\alpha} b_\alpha^L m_{\alpha}) $$

is constant and $c_1$ satisfies the property of the corollary.

\[\square\]
9. TRACE OF I_A ON THE DISCRETE SPECTRUM

Proof: For any compact open subgroup \( K' \) of \( M(\mathbf{A}) \), \( (\mathcal{A}_{M,X,cusp})^{K'} \) is the \( \mathbb{C} \)-vector subspace of the finite dimensional \( \mathbb{C} \)-vector space

\[ \{ \varphi : M(F) \backslash M(\mathbf{A}) / K' \to \mathbb{C} \mid \varphi(z_{\mathbf{A}}) = \chi(z_{\mathbf{A}}) \varphi(m_{\mathbf{A}}), \quad \forall z_{\mathbf{A}} \in Z_M(\mathbf{A}), \quad \forall m_{\mathbf{A}} \in M(\mathbf{A}), \text{ and } \text{Supp}(\varphi) \subset C_{K'} \} \]

defined by the vanishing of the \( \mathbb{C} \)-linear forms

\[ \varphi \mapsto \varphi \big|_{K'} \]

for all the proper standard parabolic subgroups \( P' \) of \( M \) (see (9.2.10) and its proof). But this last \( \mathbb{C} \)-vector space and these \( \mathbb{C} \)-linear forms are obviously defined over \( \mathbb{Q}(\chi) \). Therefore the natural map

\[ \mathbb{C} \otimes_{\mathbb{Q}(\chi)} \mathcal{A}_{M,X,cusp}^{K'} \to \mathcal{A}_{M,X,cusp}^{K'} \]

is an isomorphism.

It follows that

\[ \mathcal{A}_{M,cusp}^{\chi,gen} \overset{\text{def}}{=} \mathcal{A}(\mathbb{Q}(\chi)[X_1, \ldots, X_n] \otimes_{\mathbb{Q}(\chi)} \mathcal{A}_{M,X,cusp}) \]

is an \( M(\mathbf{A}) \)-invariant \( \mathbb{Q}(\chi) \)-structure on \( \mathcal{A}_{M,cusp}^{\chi,gen} \).

A cuspidal automorphic irreducible representation of \( M(\mathbf{A}) \) is an irreducible representation of \( M(\mathbf{A}) \) which is isomorphic to a subquotient of \( (\mathcal{A}_{M,cusp}, R_{M,cusp}) \). A cuspidal automorphic irreducible representation of \( M(\mathbf{A}) \) is obviously an automorphic irreducible representation of \( M(\mathbf{A}) \) and therefore is admissible (see (9.1.3)). The same arguments as in the proof of (9.1.16) show that a cuspidal automorphic irreducible representation of \( M(\mathbf{A}) \) is isomorphic to a subquotient of

\[ (\mathcal{A}_{M,cusp}, R_{M,cusp}) = R_{M,cusp}|_{M(\mathbf{A})} \]

for a unique \( \chi \in \mathcal{X}_M \) (\( \chi \) is its central character).

For any \( \chi \in \mathcal{X}_M \) we denote by

\[ \Pi_{M,cusp} = \mathcal{A}_{M,cusp}(M(F) \backslash M(\mathbf{A})) \]

a system of representatives of the isomorphism classes of cuspidal automorphic representations of \( M(\mathbf{A}) \) which admit \( \chi \) as a central character, i.e. which are isomorphic to a subquotient of \( (\mathcal{A}_{M,cusp}, R_{M,cusp}) \).

For each smooth irreducible representation \((\mathcal{V}, \pi)\) of \( M(\mathbf{A}) \) which admits \( \chi \) as a central character we denote by

\[ m_{\text{cusp}}(\pi) \]

the dimension of the \( \mathbb{C} \)-vector space

\[ \text{Hom}_\text{Rep}(\mathcal{A}_{M(\mathbf{A})})(\mathcal{V}, \pi)(\mathcal{A}_{M,X,cusp}, R_{M,X,cusp}) \]

(\( m_{\text{cusp}}(\pi) > 0 \) the representation \((\mathcal{V}, \pi)\) is a cuspidal automorphic one).
THEOREM (9.2.14) (Gelfand and Piatetski-Shapiro). — Let \( \chi \in \chi_M \). Then the set \( \Pi_{M, \chi, \text{cusp}} \) is countable. For each \((V, \pi) \in \Pi_{M, \chi, \text{cusp}}\) the multiplicity \( m_{\text{cusp}}(\pi) \) is finite and non-zero. The representation \((A_{M, \chi, \text{cusp}}, R_{M, \chi, \text{cusp}})\) is (non-canonically) isomorphic to

\[
\bigoplus_{(V, \pi) \in \Pi_{M, \chi, \text{cusp}}} (V, \pi)^{\otimes m_{\text{cusp}}(\pi)}.
\]

Moreover, for any compact open subgroup \( K' \) of \( M(A) \) there are only finitely many \((V, \pi) \in \Pi_{M, \chi, \text{cusp}}\) such that \( \forall K' \neq (0) \).

Proof : The representation \((A_{M, \chi, \text{cusp}}, R_{M, \chi, \text{cusp}})\) is admissible (see (9.2.10)).

If \( \mu \) is a complex character of \( \bigoplus_{d=1}^{d} \frac{1}{d} Z \) and if we set

\[
\chi' = (\mu \mid Z^d) \circ \text{deg}_Z \chi
\]

the map

\[
(A_{M, \chi, \text{cusp}}, R_{M, \chi, \text{cusp}} \otimes (\mu \circ \text{deg}_M)) \longrightarrow (A_{M', \chi', \text{cusp}}, R_{M', \chi', \text{cusp}}),
\]

\[
\varphi \longmapsto (\mu \circ \text{deg}_M) \varphi,
\]

is an isomorphism. Therefore we may assume that \( \chi \) is unitary (choose \( \mu \) such that \( \mu Z^d = |\chi|^{-1} \)).

Now for any \( \varphi_1, \varphi_2 \in A_{M, \chi, \text{cusp}} \) we set

\[
\langle \varphi_1, \varphi_2 \rangle_M = \int_{Z_M(A)(M(F)/M(A))} \frac{m_\varphi_1(m_\varphi_2)}{dZ_A dm} dm_A,
\]

\((dm_A \text{ and } dz_A \text{ are arbitrary Haar measures on } M(A) \text{ and } Z_M(A)) \text{ and } d\mu \text{ is the counting measure on } M(F); \text{ because } \chi \text{ is unitary, the integral is well-defined and, thanks to (9.2.6), it is absolutely convergent and in fact can be computed and, reducing to a finite sum). Then (*) is an } M(A) \text{-invariant, positive definite, Hermitian scalar product on } A_{M, \chi, \text{cusp}} \text{ and the theorem follows from (D.6.7) and remarks (D.6.8.1) and (D.6.8.2) (we leave it to the reader to check that } M(A) \text{ admits a countable basis of neighborhoods of } 1). \quad \Box
\]

Remark (9.2.15). — It can be proved and we will use later that \( m_{\text{cusp}}(\pi) = 1 \) for any \((V, \pi) \in \Pi_{M, \chi, \text{cusp}}\) (see [Sh], Theorem 5.5). \( \Box \)

9. TRACE OF \( f_A \) ON THE DISCRETE SPECTRUM

We will conclude this section by studying some rationality properties of cuspidal automorphic irreducible representations of \( M(A) \).

Let us fix a character \( \chi \in \chi_M \) which is trivial on \( Z_M(F) \). As \( Z_M(F) \) acts on \( M(A) \) and \( Z_M(A) \) is finite (see (9.1)) the subfield \( Q(\chi) \) of \( C \) is a number field (i.e. a finite extension of \( Q \)).

We will simply denote by

\[ \mathcal{H}_M^\chi \text{ (resp. } \mathcal{H}_M \) the \( Q(\chi) \)-algebra \( C^\infty(Z_M(F) \setminus M(A), Q(\chi)) \) (resp. \( C^\infty(Z_M(F) \setminus M(A), C) \)) of smooth functions with compact support on \( Z_M(F) \setminus M(A) \) and with values in \( Q(\chi) \) (resp. \( C \)). We have

\[ \mathcal{H}_M = C \otimes_{Q(\chi)} \mathcal{H}_M^\chi. \]

Having fixed a Haar measure \( dm_A \) on \( M(A) \) and the Haar measure \( dz_{M, \infty} \) on \( Z_M(F) \) which is normalized by \( \text{vol}(Z_M(F), dz_{M, \infty}) = 1 \), the \( C \)-algebra \( \mathcal{H}_M \) acts on the complex vector space \( A_{M, \chi, \text{cusp}} \). We will assume that \( dm_A \) is rational, i.e. that \( \text{vol}(K, dm_A) = 1 \) for some (and therefore any) compact open subgroup \( K_M \) of \( M(A) \) (see (D.1)). Then, by (9.2.11),

\[ A_{M, \chi, \text{cusp}}^\chi = \{ \varphi \in A_{M, \chi, \text{cusp}} \mid \varphi(M(A)) \subset Q(\chi) \} \]

is an \( \mathcal{H}_M^\chi \)-invariant \( Q(\chi) \)-structure on \( A_{M, \chi, \text{cusp}} \).

Let \((V, \pi) \) be a cuspidal automorphic irreducible representation of \( M(A) \) with central character \( \chi \). Let us denote by \( Q(\pi) \) the subfield of \( C \) generated by the complex numbers \( \text{tr} \pi(f) \) for \( f \in \mathcal{H}_M^\chi \). It is easy to see that \( Q(\pi) \supset Q(\chi) \) if \( z_A \) is a central element in \( M(A) \) and if \( K_M \) is a compact open subgroup of \( M(A) \) such that \( \mathcal{H}_M^K \neq (0) \).

\[ \text{tr} \pi(1_{Z_M(F)} z_A K_M) = \chi(z_A) \text{vol}(Z_M(F), K_M, dm_A) \text{dim}_C \mathcal{H}_M^K \]

is a non-zero rational multiple of \( \chi(z_A) \).

Lemma (9.2.16). — The field \( Q(\pi) \) is a number field and the isotypical component of \((V, \pi) \) in \((A_{M, \chi, \text{cusp}}, R_{M, \chi, \text{cusp}}) \) (which is non-canonically isomorphic to \((V, \pi)^{\otimes m_{\text{cusp}}(\pi)} \) by (9.2.14)) has a natural \( Q(\pi) \)-structure.

Moreover \((V, \pi) \) itself has a rational structure over a finite extension of \( Q(\pi) \).

Proof : As \((A_{M, \chi, \text{cusp}}, R_{M, \chi, \text{cusp}}) \) is admissible and admits a \( Q(\chi) \)-structure the lemma follows from general considerations (see (D.10)). \( \Box \)
DRIN Feld MODULAR VARIETIES

(9.3) $L^2$-automorphic representations

Let us fix a unitary character $\chi \in \mathcal{X}_M$. An automorphic form $\varphi \in \mathcal{A}_{M,X}$ is said to be square-integrable if

$$\int_{\mathcal{Z}(M(A)/\mathcal{M}(A))} |\varphi(m_k)|^2 \frac{dm_k}{dz \mu} < +\infty$$

($dm_k, dz \mu$ and $d\mu$ are the same as in the proof of (9.2.14)). We denote by

$$\mathcal{A}^2_{M,X} = \mathcal{A}^2_{\chi}(\mathcal{M}(F)/\mathcal{M}(A), \mathcal{C}) \subset \mathcal{A}_{M,X}$$

the C-vector subspace of the square-integrable automorphic forms. Obviously it is stable under $R_{M,X}(\mathcal{M}(A))$. We denote by $R_{M,X}^2$ the restriction of $R_{M,X}$ to $\mathcal{A}^2_{M,X}$.

The smooth representation $(\mathcal{A}^2_{M,X}, R^2_{M,X})$ of $\mathcal{M}(A)$ is unitarizable: if we set

$$\langle \varphi_1, \varphi_2 \rangle = \int_{\mathcal{Z}(M(A)/\mathcal{M}(A))} \overline{\varphi_1(m_k)} \varphi_2(m_k) \frac{dm_k}{dz \mu} \quad (\forall \varphi_1, \varphi_2 \in \mathcal{A}^2_{M,X})$$

$\langle \cdot, \cdot \rangle$ is an $\mathcal{M}(A)$-invariant, positive definite, Hermitian scalar product on $\mathcal{A}^2_{M,X}$.

**Lemma (9.3.3).** We have

$$\mathcal{A}_{M,X,\text{cusp}} \subset \mathcal{A}^2_{M,X}$$

and

$$\mathcal{A}^2_{M,X} = \mathcal{A}_{M,X,\text{cusp}} \oplus \mathcal{A}^2_{M,X,\text{Eis}}$$

where $\mathcal{A}^2_{M,X,\text{Eis}}$ is the orthogonal subspace of $\mathcal{A}_{M,X,\text{cusp}}$ in $\mathcal{A}^2_{M,X}$.

**Proof**: The first assertion has already been checked in the proof of (9.2.14). The second follows from the admissibility of $(\mathcal{A}_{M,X,\text{cusp}}, R_{M,X,\text{cusp}})$ (see (9.2.10)) and from (D.6.6).

An $L^2$-automorphic irreducible representation of $\mathcal{M}(A)$ is an irreducible representation of $\mathcal{M}(A)$ which is isomorphic to a subquotient of $(\mathcal{A}^2_{M,X}, R^2_{M,X})$ (for some unitary $\chi \in \mathcal{X}_M$). It is automatically an automorphic irreducible representation of $\mathcal{M}(A)$ and therefore it is admissible (see (9.1.3)). Then it follows from (9.1.3) and (D.6.6) that any $L^2$-automorphic irreducible representation of $\mathcal{M}(A)$ is isomorphic to an orthogonal direct summand of $(\mathcal{A}^2_{M,X}, R^2_{M,X})$ with respect to $\langle \cdot, \cdot \rangle$ for some unitary $\chi \in \mathcal{X}_M$ and therefore is unitarizable.

Any cuspidal automorphic irreducible representation of $\mathcal{M}(A)$ with unitary central character is automatically $L^2$-automorphic.

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9. TRACE OF $f_A$ ON THE DISCRETE SPECTRUM

(9.3.4) $A^2_{M,X,\text{disc}} = A^2_{\chi}(\mathcal{M}(F)/\mathcal{M}(A), \mathcal{C}) \subset A^2_{M,X}$ is the sum of all $\mathcal{M}(A)$-invariant C-vector subspaces of $\mathcal{A}^2_{M,X}$ such that $V, R^2_{M,X}(V)$ is irreducible. It is invariant under $R^2_{M,X}(\mathcal{M}(A))$. We denote by $R^2_{M,X,\text{disc}}$ the restriction of $R^2_{M,X}$ to $A^2_{M,X,\text{disc}}$.

It follows from (9.3.3) and (9.2.14) that

$$\mathcal{A}_{M,X,\text{cusp}} \subset A^2_{M,X,\text{disc}}.$$  

By the admissibility of $(\mathcal{A}_{M,X,\text{cusp}}, R_{M,X,\text{cusp}})$ (see (9.2.10)) we have

$$A^2_{M,X,\text{disc}} = \mathcal{A}_{M,X,\text{cusp}} \oplus A^2_{M,X,\text{res}}$$

(orthogonal direct sum) where we have set

$$A^2_{M,X,\text{res}} = A^2_{M,X,\text{disc}} \cap A^2_{M,X,\text{Eis}}$$

(see (D.6.6)).

For each smooth irreducible representation $(V, \pi)$ of $\mathcal{M}(A)$ which admits $\chi$ as a central character we denote by

$$m^2(\pi)$$

the dimension of the C-vector space

$$\text{Hom}_{\mathcal{R}(\mathcal{M}(A))}(V, \pi, (A^2_{M,X}, R^2_{M,X}))$$

(we have $m^2(\pi) > 0$ if and only if $(V, \pi)$ is $L^2$-automorphic). We also set

$$m^2_{\text{res}}(\pi) = m^2(\pi) - m_{\text{cusp}}(\pi).$$

We have $m^2_{\text{res}}(\pi) \geq 0$ for any $(V, \pi)$.

**Theorem (9.3.7)** (Harish-Chandra; Borel and Jacquet). — For any admissible irreducible representation $(V, \pi)$ of $\mathcal{M}(A)$ the dimension of the C-vector space

$$\text{Hom}_{\mathcal{R}(\mathcal{M}(A))}(V, \pi, (C^\infty_M, R_M))$$

is finite.